Math 656 • FINAL EXAM • May 9, 2016

1) (10pts) Find **all values** of $\cosh^{-1}(2i)$, and plot them as point in the complex plane (hint: convert to a quadratic equation for e^z)

2) (24pts) Describe all singularities of the integrand inside the integration contour, and calculate each integral (use any method you like). Each integration contour is a circle of specified radius

(a) $\oint\limits_{|z|=3} \frac{dz}{\left(z^2+1\right)^2}$ (b) $\oint\limits_{|z|=5} \frac{dz}{z^2 \cos z}$ (c) $\oint\limits_{|z|=3} \frac{dz}{\bar{z}}$ (d) $\oint\limits_{|z|=1} \exp\left(\frac{1}{z}+z\right)dz$

Hint for (d): note that $\exp\left(\frac{1}{z}+z\right)=\exp\left(\frac{1}{z}\right)\exp(z)$; multiply the two series, and express the residue as a series.

3) (10pts) Find the first **two** dominant terms in the series expansion of $f(z)=\frac{\cos z-1}{z^2\left(e^z-1\right)}$ near $z = 0$, and use your result to classify the singularity at z=0. What is the residue of this function at z=0? What would be the domain of convergence of the corresponding full series? Finally, classify the singularity of this function at z=∞, as well.

4) (10pts) Sketch the domain of convergence of the Laurent series $\sum_{k=0}^{\infty}\left[\frac{(2i+z)^k}{e^{2k}}+\frac{3^k}{k!\,(2i+z)^{2k}}\right]$, and write down the expression for its sum. What are the singularities of this sum (which represents the analytic extension of this series)? **Hint:** this is a very straightforward problem: notice a combination of standard series of familiar elementary functions.

5) (16pts) Calculate **two** of the following integrals. Explain each step briefly but fully. If you choose (c), use an “indented” contour. Make sure to obtain a real answer in each problem!

(a) $\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$ (b) $\int_{-\infty}^{+\infty} \frac{x^5 \sin(2x)\,dx}{1+x^6}$ (c) $\int_{-\infty}^{+\infty} \frac{\cos x-\cos(2x)}{x^2}dx$

6) (10pts) Use Rouche’s Theorem to find an annulus/ring with integer radii, $n < |z| < n+1$ $\left(n\in\mathbb{Z}_+\right)$, containing all roots of polynomial $f(z) = z^3 + z^2 + 40$

================== You may drop one problem out of the last three ==================

7) (10pts) Use the Argument Principle to find the number of roots of $f(z) = 2i + z + z^4$ lying in the first quadrant. To do this, sketch the mapping of the relevant quarter-circular sector boundary (it’s quite straightforward).

8) (10pts) What is the image of the domain $\left\{\operatorname{Re}(z)\in\left[0,\ \frac{\pi}{2}\right],\ \operatorname{Im}(z)\in[0,\ +\infty]\right\}$ under the mapping $w=\sin z$?

Hint: consider separately the map of each boundary, and the map of any point or curve within this domain. You may use the Cartesian decomposition $\sin z=\sin x\cosh y+i\,\cos x\sinh y$. Note that a map does not preserve angles (is not conformal) wherever $f'(z)=0$.

9) (10pts) Suppose $f(z)$ and $g(z)$ are entire functions, and that $|f(z)| \le 10\,|g(z)|$ in the entire complex plane. Is it true that $f(z) = \lambda g(z)$ for all z, where λ is a constant? If true, explain carefully, using any theorem learned in this course. If not true, give a counterexample. Note that $f(z)$ and $g(z)$ may have zeros in the complex plane.